Lesson 8 Welfare Analysis

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- Lessons 6 and 7 shows that Taylor rule with $\varphi_\pi > 1$ and exogenous money supply rule give uniqueness in new Keynesian model
- Which rule is better or superior?
- Similar to classical monetary model, new Keynesian model is derived from solving households' maximization problem.
- In addition, because all of firms are owned by households, we can discuss the merits brought about by these 2 monetary policies from the viewpoint of maximizing welfare.

15.1 Welfare Cost Function

- Welfare cost function can be derived through second-order approximation of households' utility function.
- Now, we review second-order Taylor expansion. Function $V(X_t)$'s second-order Taylor expansion around the steady state is given by:

$$V(X_t)=V(X)+V_x(X)(X_t-X)+\frac{1}{2}V_{xx}(X)(X_t-X)^2+o(\|\xi\|)^3 \quad (15.1)$$

where X_t is an arbitrary variable, X is the steady state value of X_t and $o(\|\xi\|)^3$ is terms of third or higher-order.

- Although it is tricky, X_t itself can be taken second-order approximation. Second order approximation of X_t is given by:

$$X_t=X\left(1+x_t+\frac{1}{2}x_t^2\right)+o(\|\xi\|)^3 \quad (15.2)$$

with $x_t \equiv \frac{dX_t}{X}=\ln\left(\frac{X_t}{X}\right)$.

Proof of Eq.(15.2)

- Pay attention to $X_t=\exp(\ln X_t)$ and $\partial \exp(X_t)/(\partial X_t)=\exp(X_t)$.
- The, X_t can be expanded as:

$$\begin{aligned}
X_t &= \exp(\ln x_t)\\
&=\exp(\ln x)+\exp(\ln x)(\ln X_t-X)+\frac{1}{2}\exp(\ln x)(\ln X_t-X)^2+o(\|\xi\|)^3\\
&=\exp(\ln x)\left[1+(\ln X_t-X)+\frac{1}{2}(\ln X_t-X)^2\right]+o(\|\xi\|)^3\\
&=\exp(\ln x)\left[1+\ln\left(\frac{X_t}{X}\right)+\frac{1}{2}\ln\left(\frac{X_t}{X}\right)^2\right]+o(\|\xi\|)^3\\
&=X\left(1+x_t+\frac{1}{2}x_t^2\right)+o(\|\xi\|)^3
\end{aligned}$$

QED

15.1.1 Second-order Approximation

- Suppose that the utility function is given by:

$$\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t U(C_t,N_t) \quad (14.1)$$

where $U(C_t,N_t)$ denotes the period utility which consists of component of consumption and labor. That is.

$$U(C_t,N_t)=u(C_t)+v(N_t)$$

- This is consistent with additively separable utility which has been assumed.
- First, We approximate each component and then sum up them.
- Now, we approximate component of consumption $u(C_t)$.

- Eqs.(15.1) and (15.2) yields:

$$u(C_t)=u(C)+u_C(C)\left[C\left(1+c_t+\frac{1}{2}c_t^2\right)-C\right]+\frac{1}{2}u_{CC}(C)\left[C\left(1+c_t+\frac{1}{2}c_t^2\right)-C\right]^2$$
$$+o(\|\xi\|)^3$$
$$=u_C(C)\left(Cc_t+\frac{1}{2}Cc_t^2\right)+\frac{1}{2}u_{CC}(C)\left(Cc_t+\frac{1}{2}Cc_t^2\right)^2+\text{t.i.p}+o(\|\xi\|)^3$$
$$=u_C(C)Cc_t+u_C(C)C\frac{1}{2}c_t^2+\frac{1}{2}u_{CC}(C)C^2c_t^2+\text{t.i.p}+o(\|\xi\|)^3$$
$$=U_C(C)Cc_t+C\frac{1}{2}c_t^2\left[U_C(C)+U_{CC}(C)C\right]+\text{t.i.p}+o(\|\xi\|)^3 \qquad (15.3)$$

where t.i.p denotes terms independent policy.

- t.i.p. includes terms which are not determined by monetary policy.
- Monetary policy can determines the percentage deviation of variables from their steady state value although cannot determine the level of those at the steady state. thus, t.i.p. includes steady state value of them which are constants.
- Because of this, $U(C)$ disappears from Eq.(15.4).
- t.i.p. includes exogenous variables which are not determined by monetary policy. Eq.(15.4) does not include exogenous variables at all.

- Next, we approximate component of labo $U(N_t)$.
- Similar to Eq.(15.3), combining Eqs.(15.1) and (15.2) yields:

$$v(N_t)=v_N(N)Nn_t+N\frac{1}{2}n_t^2\left[v_N(N)+v_{NN}(N)N\right]+\text{t.i.p}+o(\|\xi\|)^3 \qquad (15.4)$$

- Combining Eqs.(15.3) and (15.4) yields:

$$\frac{U(C_t,N_t)}{u_C(C)C}=c_t+\frac{v_N(N)}{u_C(C)}\frac{N}{C}n_t+\frac{1}{u_C(C)}\frac{1}{2}c_t^2\left[u_C(C)+u_{CC}(C)C\right]$$
$$+\frac{N}{C}\frac{1}{2}n_t^2\left[\frac{v_N(N)}{u_C(C)}+\frac{v_N(N)}{u_C(C)}\frac{v_{NN}(N)N}{v_N(N)}\right]+\text{t.i.p}+o(\|\xi\|)^3 \qquad (15.5)$$

15.1.2 Trade-off between consumption and Labor at the Steady State

- Because the budget constraint is still given by Eq.(14.9), intratemporal optimality condition for households is given by:

$$-\frac{v_{N_t}}{u_{C_t}}=\frac{W_t}{P_t} \qquad (15.6)$$

- In the steady state, related to the margnal cost, $MC=W(1-\tau)/P$ and $MC=(\varepsilon-1)/\varepsilon$ are applied. By plugging these into Eq.(15.5), we can understand that followings are applied in the steady state:

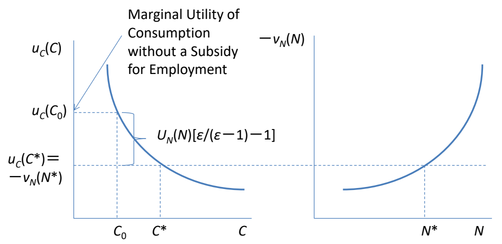
$$-\frac{v_N}{u_c}=\frac{W}{P}$$
$$=MC\frac{1}{1-\tau}$$
$$=\frac{\varepsilon-1}{\varepsilon}\frac{1}{1-\tau} \qquad (15.7)$$

- Eq.(15.7) can be rewritten as:

$$-v_N=\frac{\varepsilon-1}{\varepsilon}\frac{1}{1-\tau}u_C \qquad (15.8)$$

which shows a trade-off between consumption and labor in the steady state where τ denotes a subsidiary for employment from the government.
- Suppose that $\tau=0$.Because of $\varepsilon>1$, $(\varepsilon-1)/\varepsilon<1$ is applied and the marginal utility of consumption exceeds the marginal disutility of labor.
- In other words, consumption goes below the level in which the marginal utility of consumption equals to marginal disutility of labor (Fig. 15-1).

Fig. 15-1: Consumption in the Distorted Steady State



C^*: Consumption in which the marginal utility of consumption equals to the marginal disutility of labor
C_0: Consumption in which the marginal utility of consumption exceeds the marginal disutility of labor

- This stems from monopolistical competitive power in goods market. Firms obtain excess earnings which does not affect households' schedule on consumption.
- This can be understood by plugging τ=0 into Eq.(15.8) which gives:

$$W=\frac{\varepsilon-1}{\varepsilon}P$$

- This equality shows that the nominal wage go below the price. That is, households do not consume enough.
- This kind of steady state is dubbed distorted steady state.

- Now, we assume that this distortion is dissolved by a subsidy for employment paid for firms from the government.
- To dissolve, τ must be chosen to suffice following:

$$\frac{\varepsilon-1}{\varepsilon}\frac{1}{1-\tau}=1$$

- To suffice this, $\tau=1/\varepsilon$ must be chosen. If the government chooses $\tau=1/\varepsilon$, Eq.(15.8) can be rewritten as:

$$-v_N=u_C \tag{15.9}$$

  Under Eq.(15.9), the marginal utility of consumption equals to the marginal disutility of labor. Consumption is larger than one in the distorted steady state.
- The steady state which suffices Eq.(15.9) is dubbed undistorted steady state.

- Plugging Eq.(15.9) into Eq.(15.5) yields:

$$\frac{U(C_t,N_t)}{u_C(C)C}=c_t-\frac{N}{C}n_t+\frac{1}{2}c_t^2\left[1+\frac{u_{CC}(C)C}{u_C(C)}\right]-\frac{N}{C}\frac{1}{2}n_t^2\left[1+\frac{v_{NN}(N)N}{v_N(N)}\right]+\text{t.i.p}+o\left(\|\xi\|\right)^3 \tag{15.10}$$

- Production function is given by Eq.(14.34). That is:

$$N_t\equiv\int_0^1 N_t(j)dj=\int_0^1 Y_t(j)djA_t^{-1}=Z_tY_tA_t^{-1} \tag{15.11}$$

  where:

$$Z_t\equiv\int_0^1\left(P_t(j)/P_t\right)^{-\varepsilon}dj$$

  denotes the price dispersion. As mentioned, the price dispersion Z_t is 1 in the steady state. also, the productivity is 1. Thus, $N=Y$ is applied in the steady state. Further, because of $Y=C$, $N=C$ is applied.

- Because of $N=C$, we can rewrite Eq.(15.10) as follows:

$$\frac{U(C_t,N_t)}{u_C(C)C}=c_t-n_t+\frac{1}{2}c_t^2\left[1+\frac{u_{CC}(C)C}{u_C(C)}\right]-\frac{1}{2}n_t^2\left[1+\frac{v_{NN}(N)N}{v_N(N)}\right]+\text{t.i.p}+o\left(\|\xi\|\right)^3 \tag{15.12}$$

- Now we suppose $u(C_t)=\ln C_t$ and $-v(N_t)=1/2N_t^2$. That is, the same as before, $U(C_t,N_t)=\ln C_t-1/2N_t^2$. Then, $u_C=C^{-1}$, $u_{CC}=-C^{-2}$, $-v_N=N$ and $-v_{NN}=1$. Plugging these into Eq.(15.12) yields:

$$U(C_t,N_t)=c_t-n_t-n_t^2+\text{t.i.p}+o\left(\|\xi\|\right)^3$$

- Because of equilibrium condition Eq. (14.32), namely, $y_t=c_t$ is applied, this equality can be rewritten as:

$$U(C_t,N_t)=y_t-n_t-n_t^2+\text{t.i.p}+o\left(\|\xi\|\right)^3 \tag{15.13}$$

15.1.3 Price Dispersion

- Now, we reconsider production function Eq. (14.34).
- Lesson 4 shows that Eq.(14.34) can be log-linearized as:

$$n_t = y_t - a_t \quad (14.35)$$

- Log-linearization is one of first order approximation which ignores a second order term Z_t. Because second-order terms cannot be ignored, in second-order approximation, we have to respect second order terms. The, we have:

$$n_t = y_t - a_t + z_t \quad (15.14)$$

- note that Z=1 is applied in the steady state because all of goods' average prices are 1 (Z_t is also average price of goods).

- As shown, price dispersion Z_t is defined as:

$$Z_t \equiv \int_0^1 \left(P_t(j)/P_t\right)^{-\varepsilon} dj \quad (15.11)$$

- We take second order approximation on Eq.(15.11).
- Let rewrite $(P_t(j)/P_t)^{1-\varepsilon}$ as follows:

$$\left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} = \exp\left\{\ln\left[\left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon}\right]\right\} = \exp\left[(1-\varepsilon)\hat{p}_t(j)\right]$$

with $\hat{p}_t(j) \equiv \ln P_t(j) - \ln P_t$.

- Now we confirm followings:

$$\frac{\partial \exp\left[(1-\varepsilon)\hat{p}_t\right]}{\partial \hat{p}_t} = (1-\varepsilon)\exp\left[(1-\varepsilon)\hat{p}_t\right]$$

$$\frac{\partial \exp\left[(1-\varepsilon)\hat{p}_t\right]}{\partial\partial \hat{p}_t} = (1-\varepsilon)^2\exp\left[(1-\varepsilon)\hat{p}_t\right]$$

$$\hat{p} = 0$$

$$\exp(0) = 1$$

- Thus, second-order approximation of Eq.(15.11) is given by:

$$\left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} = 1 + (1-\varepsilon)\hat{p}_{H,t}(j) + \frac{(1-\varepsilon)^2}{2}\hat{p}_{H,t}(j)^2 + o\left(\|\xi\|\right)^3 \quad (15.14)$$

- Eq.(14.7) $P_t \equiv \left[\int_0^1 P_t(j)^{1-\varepsilon} dj\right]^{\frac{1}{1-\varepsilon}}$ can be rewritten as:

$$1 = \frac{1}{P_t}\left[\int_0^1 P_t(j)^{1-\varepsilon} dj\right]^{\frac{1}{1-\varepsilon}}$$

- Raising 1−ε th power on both sides of the previous expression yields:

$$1^{1-\varepsilon} = \left(\frac{1}{P_t}\right)^{1-\varepsilon}\left[\int_0^1 P_t(j)^{1-\varepsilon} dj\right] = \left[\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} dj\right] = 1$$

- Thus, we have:

$$\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} dj = E_j\left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} = 1 \quad (15.15)$$

That is, the average of the deviation from aggregated price is unity.

- By taking expectation on Eq.(15.14), we get:

$$E_j\left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon} = 1 + E_j\left[(1-\varepsilon)\hat{p}_{H,t}(j) + \frac{(1-\varepsilon)^2}{2}\hat{p}_{H,t}(j)^2\right] + o\left(\|\xi\|\right)^3$$

- As shown is Eq.(15.15), $E_j\left(P_t(j)/P_t\right)^{1-\varepsilon} = 1$ is applied. Thus, this can be rewritten as:

$$E_j\left(\hat{p}_{H,t}(j)\right) = -\frac{1-\varepsilon}{2}E_j\left[\hat{p}_{H,t}(j)^2\right] + o\left(\|\xi\|\right)^3 \quad (15.16)$$

- Next, we take second-order approximation on $(P_t(j)/P_t)^{-\varepsilon}$. This is given by:

$$\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} = \exp[-\varepsilon \hat{p}_t(j)]$$
$$= 1 - \varepsilon \hat{p}_t(j) + \frac{\varepsilon^2}{2}\hat{p}_t(j)^2 + o(\|\xi\|)^3$$

- Taking conditional expectation on this yields:15.17)

$$\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} dj = 1 - \varepsilon \mathrm{E}_j[\hat{p}_t(j)] + \frac{\varepsilon^2}{2}\mathrm{E}_j[\hat{p}_t(j)^2] + o(\|\xi\|)^3$$

- Here:

$$\mathrm{E}_j\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} dj = Z_t$$

is applied.

- Plugging Eq.(15.16) into the second term in the RHS in Eq.(15.17) yields:

$$\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} dj = 1 + \frac{(1-\varepsilon)\varepsilon}{2}\mathrm{E}_j[\hat{p}_{H,t}(j)^2] + \frac{\varepsilon^2}{2}\mathrm{E}_j[\hat{p}_t(j)^2] + o(\|\xi\|)^3$$
$$= 1 + \frac{\varepsilon}{2}\mathrm{E}_j[\hat{p}_{H,t}(j)^2]$$
$$= 1 + \mathrm{var}_i[\hat{p}_{H,t}(j)]$$

- The LHS of this is Z_t. The, we have:

$$z_t = \ln\left\{1 + \frac{\varepsilon}{2}\mathrm{var}_j[\hat{p}_t(j)]\right\}$$
$$\approx \frac{\varepsilon}{2}\mathrm{var}_j[\hat{p}_t(j)] \qquad (15.18)$$

- Plugging Eqs.(15.14) and (15.18) into Eq.(15.13) yields:

$$U(C_t, N_t) = y_t - (y_t - a_t + z_t) - n_t^2 + \text{t.i.p} + o(\|\xi\|)^3$$
$$= -\frac{\varepsilon}{2}\mathrm{var}_j[\hat{p}_t(j)] - n_t^2 + \text{t.i.p} + o(\|\xi\|)^3 \qquad (15.19)$$

- Here, exogenous shock a_t is depending on monetary policy and is included in t.i.p.
- Except for t.i.p., first order terms are completely disappears from Eq.(15.21).
- To calculate welfare, eliminating first order term is very important. Second-order approximated welfare costs function with first order term cannot calculate welfare correctly and generates 'welfare reversal' is well known (Kim and Kim, 1997).

15.1.4 Inflation and GDP Gap

- Here we consider the second term in RHS in Eq.(19.19) n_t^2 .
- Plugging Eq.(14.40) $y_t = \tilde{y}_t + \overline{y}_t$ and Eq.(14.42) $\overline{y}_t = a_t$ into Eq.(15.14) yields:

$$n_t = \tilde{y}_t + z_t$$

- By squaring both sides of this yields:

$$n_t^2 = (\tilde{y}_t + z_t)^2$$
$$= \tilde{y}_t^2 + 2\tilde{y}_t z_t + z_t^2$$
$$= \tilde{y}_t^2 + o(\|\xi\|)^3 \qquad (15.20)$$

Because Z_t is second order, $2\tilde{y}_t z_t$, z_t^2 are third or higher order.

- Plugging Eq.(15.20) into Eq.(15.19) yields:

$$U(C_t, N_t) = -\frac{\varepsilon}{2}\mathrm{var}_j[\hat{p}_t(j)] - \tilde{y}_t^2 + \text{t.i.p} + o(\|\xi\|)^3$$

- By calculating sum of net present value from period 0 to period infinity, households' utility function Eq.(14.1) is given by:

$$\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t U(C_t, N_t) = -\frac{1}{2}\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\varepsilon\,\mathrm{var}_j[\hat{p}_t(j)] + \tilde{y}_t^2\right\} + \text{t.i.p} + o(\|\xi\|)^3 \qquad (15.21)$$

- Woodford (2003) shows that:

$$\sum_{t=0}^{\infty}\beta^t \,\text{var}_j\left[\hat{p}_t(j)\right]=\frac{1}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$$

which implies that sum of variance of each price's relative price corresponds to the net present value of inflation multiplied by the inverse of the slope of the NKPC $(2\kappa)^{-1}$.

- Plugging this into Eq.(15.21) yields:

$$\text{E}_0\sum_{t=0}^{\infty}\beta^t U(C_t,N_t)=-\frac{1}{2}\text{E}_0\sum_{t=0}^{\infty}\beta^t\left(\frac{\varepsilon}{2\kappa}\pi_t^2+\tilde{y}_t^2\right)+\text{t.i.p}+o\left(\|\xi\|\right)^3 \quad (15.22)$$

- Eq.(15.22) shows that households' utility is decreasing function of squared inflation and GDP gap.
- In New Keynesian model, nominal rigidities are introduced as Friction. As mentioned, price stickiness generates inflation through fluctuation in the marginal cost. Further, fluctuation in the marginal costs varies GDP gap.
- Fluctuation in inflation and the GDP gap varies households' consumption schedule and decreases consumption. This decrease in consumption decreases households' utility.
- Thus, households' utility is decreasing function of squared inflation and GDP gap.
- It can be said that the role of central bank is stabilizing both inflation and GDP gap.

- Welfare criteria in this economy is derived from Eq.(15.22).
- Now, we focus on the first term in the RHS in Eq.(15.22) and impose β→1. In that case, we get:

$$\text{L}=\frac{1}{2}\left(\frac{\varepsilon}{2\kappa}\text{var}(\pi_t)+\text{var}(\tilde{y}_t)\right) \quad (15.22)$$

where L denotes welfare costs.

- By using Eq.(15.22), we rank monetary policy.
- From the viewpoint of minimizing welfare costs, monetary policy which minimizes volatility on inflation and GDP gap is superior.

15.2 Comparing Welfare Costs

- Now, we compare welfare costs brought about by Taylor rule with one brought about by money supply rule.
- For Taylor rule, we consider not only the case of $\varphi_\pi=1.5$ but also the case of $\varphi_\pi=5$.
- Figs. 15-2 and 15-3 shows IRFs under Taylor rule with $\varphi_\pi=1.5$ and $\varphi_\pi=5$.
- Inflation is aggressively stabilized and the GDP gap is also stabilized.

Fig.15-2: IRFs to Monetary Policy Shock under Taylor Rule ($\varphi_\pi=1.5$)

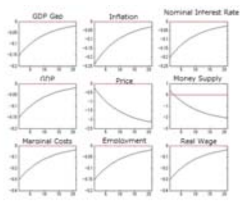
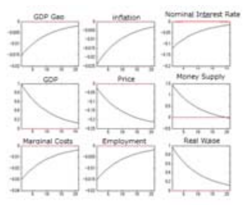


Fig.15-3: IRFs to Monetary Policy Shock under Taylor Rule ($\varphi_\pi=5$)



- Productivity shock increases productivity under both Taylor and money supply rules.
- money supply shock increases the nominal interest rate 1% under Taylor rule while it increases growth rate of money supply increases approximately 1.56%(1.555452% for correct).
- Money supply shock is different in its definition between Taylor rule and money supply rule. However, because money supply shock under money supply rule increases the nominal interest rate 1% immediately, we can compare money supply shock with Taylor rule as long as we adopt this setting.

Tab.15-1: Comparing Welfare Costs

		Taylor Rule		Money Supply Rule
		Φ_π=1.5	Φ_π=5	
Volatility	GDP Gap	24.10	0.12	20.61
	inflation	17.59	0.30	15.17
Welfare Costs		556.31	2.88	354.94

- We set the elasticity of substitution among goods ε to 7.8 and calculate welfare cost (Tab.15-1).
- Coefficient of squared inflation in the welfare cost function ε/(2κ) is approximately 45.44. That is, welfare cost is mostly shared by variance of inflation.
- This is consistent with the fact that the friction is price stickiness which generates inflation in this economy.
- Thus, monetary policy which stabilizes inflation more is evaluated.

Tab.15-1: Comparing Welfare Costs

		Taylor Rule		Money Supply Rule
		Φ_π=1.5	Φ_π=5	
Volatility	GDP Gap	24.10	0.12	20.61
	inflation	17.59	0.30	15.17
Welfare Costs		556.31	2.88	354.94

- There are no notable difference on variance of inflation between money supply rule and Taylor rule (φ_π＝1.5).
- Under Taylor rule (φ_π＝5), variance of inflation is 0.30 and is extremely low.
- Because of this, welfare cost is also extremely low under Taylor rule (φ_π＝5).

- This comparison does not necessarily conclude that Taylor rule is superior to money supply rule.
- However, it can be said that the higher the coefficient on Taylor rule φ_π, the more stabilizing inflation and the smaller welfare costs.